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A Method of Solution for a System of Two Second-Order Ordinary Differential Equations Arising in the Theory of the Mean Atmospheric Waves

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ABSTRACT

A detailed method of solution for a system of two second-order ordinary differential equations which arise in the steady state quasi-geostrophic theory of global monsoons or axially asymmetric circulations, is given. Two particular solutions which are very satisfactory are also shown as examples.

1. INTRODUCTION

In dealing with the dynamical influence of the Budyko-type [1] diabatic heating on the stationary harmonics of the atmospheric motion, in a quasi-geostrophic model, we are led to the following coupled system of second-order ordinary differential equations, along with their boundary conditions; c.f. Döös [2]:

$$S_1 \frac{d^2v_1}{d\xi^2} + S_2 \frac{dv_1}{d\xi} + S_3 \,.\, v_1 = S_4 \,.\, T_1 - S_5 \left(\frac{dv_2}{d\xi}\right)_s ; \tag{1}$$

$$\frac{dv_1}{d\xi} + G_1 \,.\, v_1 = H_1 \,.\, T_1 - K_1 \left(\frac{dv_2}{d\xi}\right)_s \text{ at } \xi = \xi_T ; \tag{2}$$

$$\frac{dv_1}{d\xi} + L_1 \,.\, v_1 + \mu \,.\, v_2 = M_1 \,.\, T_1 - N_1 \left(\frac{dv_2}{d\xi}\right)_s \text{ at } \xi = \xi_s ; \tag{3}$$

$$R_1 \frac{d^2v_2}{d\xi^2} + R_2 \frac{dv_2}{d\xi} + P_3 \,.\, v_2 = R_4 \,.\, T_2 + R_5 \left(\frac{dv_1}{d\xi}\right)_s ; \tag{4}$$

$$\frac{dv_2}{d\xi} + G_2 \,.\, v_2 = H_2 \,.\, T_2 + K_2 \left(\frac{dv_1}{d\xi}\right)_s \text{ at } \xi = \xi_T ; \tag{5}$$

$$\frac{dv_2}{d\xi} + L_2 \,.\, v_2 - \mu \,.\, v_1 = M_2 \,.\, T_2 + N_2 \left(\frac{dv_1}{d\xi}\right)_s \text{ at } \xi = \xi_s . \tag{6}$$

Here v_1 and v_2 are dependent variables and ξ is the independent variable. S_1, S_2, S_3, S_4, S_5, R_1, R_2, R_3, R_4, and R_5 are continuous functions of ξ. G_1, H_1, T_1, K_1, L_1, μ., M_1, N_1, G_2, H_2, T_2, K_2, L_2, M_2, N_2, are known constants. Equations (2) and (3) are the boundary conditions of (1), while (5) and (6) are the boundary conditions of (4). Subscripts T and δ indicate top and bottom (ξ_T and ξ_δ) of the atmosphere.

In applying finite-difference methods to solve the above system, one is most likely to think first in terms of some iterative procedure to obtain a solution. It is the purpose of this note to describe a variation of a more powerful method discussed by Richtmyer [3] which does not involve the problems of convergence usually associated with iterative procedures.

2. The Finite Difference Method

A. *The Grid*

To express differentials in finite difference form, we divide the region ξ_T to ξ_δ into a number of equally spaced grid intervals. The grid points will be referenced by the index j ranging from $j = 1$ at $\xi = \xi_T$ to $J = M$ at $\xi = \xi_\delta$. Also we make use of two fictitious points, $J = 0$ and $j = M + 1$, on either side of this range, to apply the boundary conditions. The grid distance between two consecutive points will be denoted by $\delta\xi$. We shall use centered-difference formulas for derivatives.

B. *The Body Equations*

Thus, in finite-difference form, we can write for (1) and (4),

$$\alpha_1(j) \,.\, v_1(j+1) + \beta_1(j) \,.\, v_1(j) + \gamma_1(j) \,.\, v_1(j-1) = \chi_1(j) \,.\, T_1 - \psi_1(j) \,.\, (\Delta v_2/\Delta\xi)_\delta , \tag{7}$$

$$\alpha_2(j) \,.\, v_2(j+1) + \beta_2(j) \,.\, v_2(j) + \gamma_2(j) \,.\, v_2(j-1) = \chi_2(j) \,.\, T_2 + \psi_2(j) \,.\, (\Delta v_1/\Delta\xi)_\delta , \tag{8}$$

where

$$\begin{aligned}
\alpha_1(j) &= [2S_1(j) + S_2(j) \,.\, \delta\xi], \\
\beta_1(j) &= [2S_3(j) \,.\, \partial\xi^2 - 4S_1(j)], \\
\gamma_1(j) &= [2S_1(j) - S_2(j) \,.\, \delta\xi], \\
\chi_1(j) &= S_4(j) \,.\, 2 \,.\, \delta\xi^2, \qquad \psi_1(j) = S_5(j) \,.\, 2 \,.\, \delta\xi^2, \\
\left(\frac{\Delta v_2}{\Delta\xi}\right)_\delta &= \frac{v_1(M+1) - v_2(M-1)}{2\delta\xi}, \\
\alpha_2(j) &= [2R_1(j) + R_2(j) \,.\, \delta\xi], \\
\beta_2(j) &= [2R_3(j) \,.\, \delta\xi^2 - 4R_1(j)], \\
\gamma_2(j) &= [2R_1(j) - R_2(j) \,.\, \delta\xi], \\
\chi_2(j) &= R_4(j) \,.\, 2 \,.\, \delta\xi^2, \qquad \psi_2(j) = R_5(j) \,.\, 2 \,.\, \delta\xi^2, \\
\left(\frac{\Delta v_1}{\Delta\xi}\right)_\delta &= \frac{v_1(M+1) - v_1(M-1)}{2\delta\xi}.
\end{aligned}$$

Now let us seek a two parameter family of solutions of (7) and (8) in the form

$$V_1(j) = E_1(j) \, . \, V_1(j+1) + F_1(j) + \phi_1(j) \, . \, (\Delta V_2/\Delta\xi)_8 \,, \tag{9}$$

$$V_2(j) = E_2(j) \, . \, V_2(j+1) + F_2(j) + \phi_2(j) \, . \, (\Delta V_1/\Delta\xi)_8 \,. \tag{10}$$

Substituting for $v_1(j-1)$ and $v_2(j-1)$ in (7) and (8) from (9) and (10), we arrive at the following relations:

$$E_1(j) = -\frac{\alpha_1(j)}{[\beta_1(j) + \gamma_1(j) \, . \, E_1(j-1)]} \, ; \tag{11}$$

$$F_1(j) = -\frac{[\gamma_1(j) \, . \, F_1(j-1) - \chi_1(j) \, . \, T_1]}{[\beta_1(j) + \gamma_1(j) \, . \, E_1(j-1)]} \, ; \tag{12}$$

$$\phi_1(j) = -\frac{[\gamma_1(j) \, . \, \phi_1(j-1) + \psi_1(j)]}{[\beta_1(j) + \gamma_1(j) \, . \, E_1(j-1)]} \, ; \tag{13}$$

$$E_2(j) = -\frac{\alpha_2(j)}{[\beta_2(j) + \gamma_2(j) \, . \, E_2(j-1)]} \, ; \tag{14}$$

$$F_2(j) = -\frac{[\gamma_2(j) \, . \, F_2(j-1) - \chi_2(j) \, . \, T_2]}{[\beta_2(j) + \gamma_2(j) \, . \, E_2(j-1)]} \, ; \tag{15}$$

$$\phi_2(j) = -\frac{[\gamma_2(j) \, . \, \phi_2(j-1) + \psi_2(j)]}{[\beta_2(j) + \gamma_2(j) \, . \, E_2(j-1)]} \, . \tag{16}$$

C. *Application of the End Conditions at* $\xi = \xi_T$

We shall first apply (7) and (8) at $j = 1$. Then we get

$$V_1(0) = \Pi_1 \, . \, T_1 + \Pi_2 \, . \, (\Delta V_2/\Delta\xi)_8 + \Pi_3 \, . \, V_1(2) + \Pi_4 \, . \, V_1(1), \tag{17}$$

$$V_2(0) = \zeta_1 \, . \, T_2 + \zeta_2 \, . \, (\Delta V_1/\Delta\xi)_8 + \zeta_3 \, . \, V_2(2) + \zeta_4 \, . \, V_2(1), \tag{18}$$

where

$$\Pi_1 = \chi_1(1)/\gamma_1(1), \qquad \Pi_2 = -\psi_1(1)/\gamma_1(1),$$

$$\Pi_3 = -\alpha_1(1)/\gamma_1(1), \qquad \Pi_4 = -\beta_1(1)/\gamma_1(1),$$

$$\zeta_1 = \chi_2(1)/\gamma_2(1), \qquad \zeta_2 = +\psi_2(1)/\gamma_2(1),$$

$$\zeta_3 = -\alpha_2(1)/\gamma_2(1), \qquad \zeta_4 = -\beta_2(1)/\gamma_2(1).$$

Now using (2), (5), (17), and (18) and applying (11) to (16), we get the following relations:

$$E_1(1) = -\frac{[1 - \Pi_3]}{[G_1 . 2\delta\xi - \Pi_4]};$$

$$F_1(1) = \frac{[H_1 . 2\delta\xi + \Pi_1]}{[G_1 . 2\delta\xi - \Pi_4]} . T_1;$$

$$\phi_1(1) = \frac{[\Pi_2 - K_1 . 2\delta\xi]}{[G_1 . 2\delta\xi - \Pi_4]};$$

$$E_2(1) = -\frac{[1 - \zeta_3]}{[G_2 . 2\delta\xi - \zeta_4]};$$

$$F_2(1) = \frac{[H_2 . 2\delta\xi + \zeta_1]}{[G_2 . 2\delta\xi - \zeta_4]} . T_2;$$

$$\phi_2(1) = \frac{[\zeta_2 + K_2 . 2\delta\xi]}{[G_2 . 2\delta\xi - \zeta_4]} .$$

Thus, after calculating $E1(1)$, $F1(1)$, $FG1(1)$, $E2(1)$, $F2(1)$, and $FG2(1)$, we can calculate $E1(j)$, $F1(j)$, $FG1(j)$, $E2(j)$, $F2(j)$, $FG2(j)$ using (11)–(16) for $j = 2$–M.

D. *Application of End Conditions at* $\xi = \xi_\delta$

We shall first apply the following conditions at $j = M$:

$$\frac{dv_1}{d\xi} = \left(\frac{dv_1}{d\xi}\right)_\delta; \qquad \frac{dv_2}{d\xi} = \left(\frac{dv_2}{d\xi}\right)_\delta. \tag{19}$$

Applying (19), using (9) and (10), we get at $j = M$:

$$\left(\frac{\Delta v_1}{\Delta\xi}\right)_\delta = P_1 . v_1(M) + P_2 + P_3 . \left(\frac{\Delta v_2}{\Delta\xi}\right)_\delta; \tag{20}$$

$$\left(\frac{\Delta v_2}{\Delta\xi}\right)_\delta = Q_1 . v_2(M) + Q_2 + Q_3 . \left(\frac{\Delta v_1}{\Delta\xi}\right)_\delta, \tag{21}$$

where

$$P_1 = \left[\frac{1}{2\delta\xi \,.\, E_1(M)} - \frac{E_1(M-1)}{2\delta\xi}\right],$$

$$P_2 = -\left[\frac{F_1(M)}{2\delta\xi \,.\, E_1(M)} + \frac{F_1(M-1)}{2\delta\xi}\right],$$

$$P_3 = -\left[\frac{\phi_1(M)}{2\delta\xi \,.\, E_1(M)} + \frac{\phi_1(M-1)}{2\delta\xi}\right],$$

$$Q_1 = \left[\frac{1}{2\delta\xi \,.\, E_2(M)} - \frac{E_2(M-1)}{2\delta\xi}\right],$$

$$Q_2 = -\left[\frac{1}{2\delta\xi \,.\, E_2(M)} + \frac{F_2(M-1)}{2\delta\xi}\right],$$

$$Q_3 = -\left[\frac{\phi_2(M)}{2\delta\xi \,.\, E_2(M)} + \frac{\phi_2(M-1)}{2\delta\xi}\right].$$

From (20) and (21) we get

$$\left(\frac{\Delta v_1}{\Delta\xi}\right)_\delta = X_1 \,.\, v_1(M) + X_2 \,.\, v_2(M) + X_3\,, \tag{22}$$

$$\left(\frac{\Delta v_2}{\Delta\xi}\right)_\delta = Y_1 \,.\, v_2(M) + Y_2 \,.\, v_1(M) + Y_3\,, \tag{23}$$

where

$$X_1 = \frac{P_1}{1 - P_3 \,.\, Q_3}, \qquad X_2 = \frac{P_3 \,.\, Q_1}{1 - P_3 \,.\, Q_3},$$

$$X_3 = \frac{P_2 + P_3 \,.\, Q_2}{1 - P_3 \,.\, Q_3}, \qquad Y_1 = \frac{Q_1}{1 - Q_3 \,.\, P_3},$$

$$Y_2 = \frac{Q_3 \,.\, P_1}{1 - Q_3 \,.\, P_3}, \qquad Y_3 = \frac{Q_2 + Q_3 - P_2}{1 - Q_3 \,.\, P_3}$$

Now we shall use (11) to (16) along with (22) and (23) and apply (3) and (6) at $j = M$. Thus we get

$$W_1 \,.\, V_1(M) + W_2 \,.\, V_2(M) + W_3 = 0, \tag{24}$$

$$Z_1 \,.\, V_2(M) + Z_2 \,.\, V_1(M) + Z_3 = 0, \tag{25}$$

where

$$W_1 = X_1 + L_1 + N_1 \,.\, Y_2 ,$$
$$W_2 = X_2 + \mu + N_1 \,.\, Y_1 ,$$
$$W_3 = X_3 - M_1 \,.\, T_1 + N_1 \,.\, Y_3 ,$$
$$Z_1 = Y_1 + L_2 - N_2 \,.\, X_2 ,$$
$$Z_2 = Y_2 - \mu - N_2 \,.\, X_1 ,$$
$$Z_3 = Y_3 - M_2 \,.\, T_2 - N_2 \,.\, X_3 .$$

From (24) and (25) we get

$$V_1(M) = \frac{W_2 \,.\, Z_3 - W_3 \,.\, Z_1}{Z_1 \,.\, W_1 - W_2 \,.\, Z_2}, \tag{26}$$

$$V_2(M) = \frac{-W_3 - W_1 \,.\, V_1(M)}{W_2}. \tag{27}$$

After obtaining $v_1(M)$ and $v_2(M)$, we can then use (9) and (10) with (22) and (23) to calculate $v_1(j)$ and $v_2(j)$ for $j = (M - 1)$ to 1. In this way we arrive at the complete solution in two scans only.

3. Some Special Results

The numerical scheme given in the last section was used to calculate the steady-state meridional wind v, due to the vertical flux of sensible heat to the atmosphere from the ground. This heat flux is associated with a uniform zonal current $U(\xi)$ flowing over an axially-asymmetric ground temperature distribution, T_g. The model used is a linearized quasi-geostrophic model similar to that described by Döös [2]. The heating is assumed to be proportional to the difference between ground temperature and the air surface temperature, while it varies according to a power law with respect to ξ. In this problem, the system (1)–(6) arises when we expand $v^* = v \sin \theta$ and T_g as follows,

$$v^* = \sum_{n=1}^{\infty} \sum_{m=1}^{n} (V_1 \cos m\lambda + V_2 \sin m\lambda)\, P_n{}^m(\cos \theta),$$

$$T_g = \sum_{n=1}^{\infty} \sum_{m=1}^{n} (T_1 \cos m\lambda + T_2 \sin m\lambda)\, P_n{}^m(\cos \theta),$$

where $V_1(\xi)$ and $V_2(\xi)$ and T_1 and T_2 are referred to in (1)–(6), and $P_n^m(\cos\theta) =$ associated Legendre polynomial of order m and degree n,

$$\theta = \text{colatitude},$$

$$m = \text{order of the Legendre polynomial},$$

$$n = \text{degree of the Legendre polynomial}.$$

Also, the coefficients S_3 and R_3 of (1)–(4) are related to n, the degree of Legendre polynomial, by the relation

$$S_3 = R_3 = |\, C_0 - n(n+1)|,$$

where C_0 is a function of ξ. Hence, for a representative value of n, we can expect quasi-resonant solutions for the system (1)–(6). In crossing this quasi-resonant point, the solutions exhibit a sudden change in character. As examples we show in

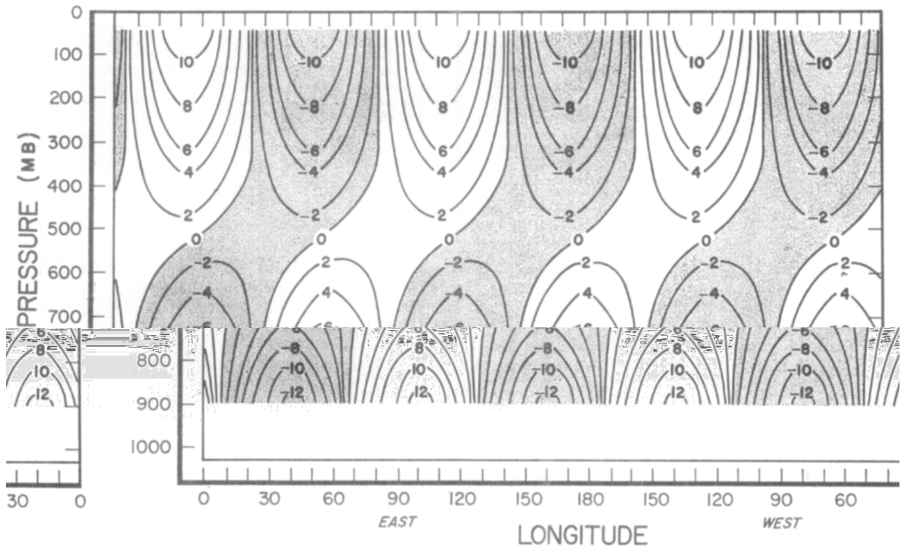

FIG. 1. Solution for $v^*(\lambda, p)$ along 45°N latitude for $(m, n) = (3, 3)$. Units are cm sec^{-1}.

Figs. 1 and 2 the v^*-solutions portrayed as cross-sections along 45°N for $(m, n) =$ (3, 3) and $(m, n) = (3, 8)$, respectively. These modes are on either side of a quasi-resonant mode $(m, n) = (3, 6)$. In obtaining these solutions, all coefficient occurring in (1) to (6) were calculated using winter climatological data at 45°N latitude. The solution obtained by Döös [2] shows a close resemblance to the solution for $(m, n) = (3, 8)$, shown in Fig. 2.

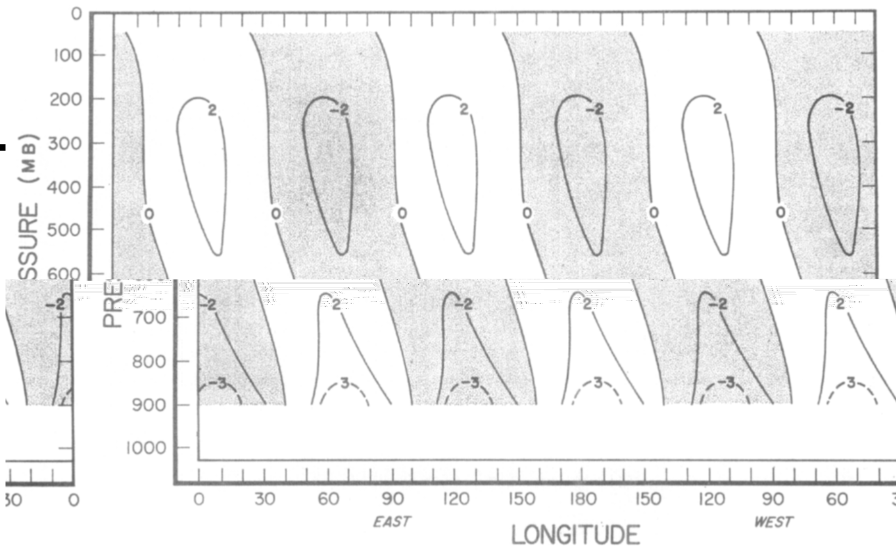

FIG. 2. Solution for $v^*(\lambda, p)$ along 45°N latitude for $(m, n) = (3, 8)$. Units are cm sec^{-1}.

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